

# Regularity of extremal solutions of semilinear fourth-order elliptic problems with general nonlinearities

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## Abstract

We consider the fourth order problem  $\Delta^2 u = \lambda f(u)$  on a general bounded domain  $\Omega$  in  $R^n$  with the Navier boundary condition  $u = \Delta u = 0$  on  $\partial\Omega$ . Here,  $\lambda$  is a positive parameter and  $f : [0, a_f) \rightarrow \mathbb{R}_+$  ( $0 < a_f \leq \infty$ ) is a smooth, increasing, convex nonlinearity such that  $f(0) > 0$  and which blows up at  $a_f$ . Let

$$0 < \tau_- := \liminf_{t \rightarrow a_f} \frac{f(t)f''(t)}{f'(t)^2} \leq \tau_+ := \limsup_{t \rightarrow a_f} \frac{f(t)f''(t)}{f'(t)^2} < 2.$$

We show that if  $u_m$  is a sequence of semistable solutions correspond to  $\lambda_m$  satisfy the stability inequality

$$\sqrt{\lambda_m} \int_{\Omega} \sqrt{f'(u_m)} \phi^2 dx \leq \int_{\Omega} |\nabla \phi|^2 dx, \quad \text{for all } \phi \in H_0^1(\Omega),$$

then  $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$  for  $n < \frac{4\alpha_*(2-\tau_+)+2\tau_+}{\tau_+} \max\{1, \tau_+\}$ , where  $\alpha^*$  is the largest root of the equation

$$(2 - \tau_-)^2 \alpha^4 - 8(2 - \tau_+) \alpha^2 + 4(4 - 3\tau_+) \alpha - 4(1 - \tau_+) = 0.$$

In particular, if  $\tau_- = \tau_+ := \tau$ , then  $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$  for  $n \leq 12$  when  $\tau \leq 1$ , and for  $n \leq 7$  when  $\tau \leq 1.57863$ . These estimates lead to the regularity of the corresponding extremal solution  $u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ , where  $\lambda^*$  is the extremal parameter of the eigenvalue problem.

*Key words:* Biharmonic, Extremal solution; Regularity of solutions.

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## 1. Introduction and main results

In this article, we consider the problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & x \in \Omega, \\ u = \Delta u = 0 & x \in \partial\Omega, \end{cases} \quad (N_\lambda)$$

where  $\Omega \subset \mathbb{R}^n$  is a smooth bounded domain,  $n \geq 1$ ,  $\lambda > 0$  is a real parameter, and the nonlinearity  $f$  satisfies

(H)  $f : [0, a_f] \rightarrow \mathbb{R}_+$  ( $0 < a_f \leq \infty$ ) is a smooth, increasing, convex function such that  $f(0) > 0$  and  $\lim_{t \rightarrow a_f} f(t) = \infty$ . Also, when  $a_f = \infty$  we assume that  $f$  is superlinear, i.e.,  $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \infty$ .

We call the nonlinearity  $f$  *regular* if  $a_f = \infty$  and *singular* when  $a_f < \infty$ .

By a semistable solution of  $N_\lambda$  we mean a solution  $u$  satisfies

$$\int_{\Omega} (\Delta \varphi)^2 - \int_{\Omega} \lambda f(u)' \varphi^2 \geq 0, \quad \varphi \in H^2(\Omega) \cap H_0^1(\Omega). \quad (1.1)$$

Also, we say that a smooth solution  $u$  of  $N_\lambda$  is minimal provided  $u \leq v$  a.e. in  $\Omega$  for any solution  $v$  of  $N_\lambda$  (see [6, 7]).

When  $f$  satisfies (H) is a regular, or  $f(t) = (1 - t)^{-p}$  ( $p > 1$ ), it is well known [2, 3, 17] that there exists a finite positive extremal parameter  $\lambda^* > 0$  depending on  $f$  and  $\Omega$  such that for any  $0 < \lambda < \lambda^*$ , problem  $(N_\lambda)$  has a minimal smooth solution  $u_\lambda$ , which is semistable and unique among the semistable solutions, while no solution exists for  $\lambda \geq \lambda^*$ . The function  $\lambda \rightarrow u_\lambda$  is strictly increasing on  $(0, \lambda_*)$ , the increasing pointwise limit  $u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$  is called the extremal solution. For  $0 < \lambda < \lambda_*$  the minimal solution  $u_\lambda$  of problem  $(N_\lambda)$  satisfies the following stability inequality, for the proof see Corollary 1 in [7] or Lemma 6.1 in [11],

$$\sqrt{\lambda} \int_{\Omega} \sqrt{f'(u_\lambda)} \phi^2 dx \leq \int_{\Omega} |\nabla \phi|^2 dx, \quad (1.2)$$

for all  $\phi \in H_0^1(\Omega)$ .

The regularity and properties of the extremal solutions have been studied extensively in the literature [2-12, 15, 19] and it is shown that it depends strongly on the dimension  $n$ , domain  $\Omega$  and nonlinearity  $f$ .

Cowan, Esposito and Ghoussoub in [6] showed that for general nonlinearity  $f$  satisfies (H),  $u^*$  is bounded for  $n \leq 5$ . When  $f(u) = e^u$ , in [6] it is shown that  $u^*$  is bounded for  $n \leq 8$ . This result improved by Cowan and Ghoussoub to  $n \leq 10$  in [7], and by Dupaigne, Ghergu and Warnault in [11] to  $n \leq 12$  which is the optimal dimension as we know on the unit ball  $u^*$  is bounded if and only if  $n \leq 12$ . As we shall see, in this paper we prove the same for a large class of nonlinearities including  $e^u$ . When  $f(u) = (1 + u)^p$  ( $p > 1$ ) in [6] it is proved that  $u^*$  is bounded if  $n < \frac{8p}{p-1}$  that improved in [7] for to  $n < 4h(p) > \frac{8p}{p-1}$  (for the definition of  $h(p)$  which is a

decreasing function on  $(1, \infty)$  see [7]) with  $\lim_{p \rightarrow \infty} 4h(p) \approx 10.718$ . Recently, Harrabi and Ye in [18] improved this result by showing that  $u^*$  is bounded for any  $p > 1$  and  $n \leq 12$ .

For the singular nonlinearity  $f(u) = (1 - u)^{-p}$  ( $p > 1$ ), in [6] it is proved that  $\sup_{\Omega} u^* < 1$  if  $n \leq \frac{8p}{p+1}$ . In particular, when  $p = 2$ ,  $u^*$  is bounded away from 1 for  $n \leq 5$ . The later result (and also the general case  $1 < p \neq 3$ ) is improved in [7] to  $n \leq 6$ , and further improved by Guo and Wei in [14] to  $n \leq 7$ . However, for  $p = 2$  the expected optimal dimension is  $n = 8$ , holds on the ball, see [15].

By imposing extra assumptions on the general nonlinearity  $f$  satisfies (H), the authors in [6] obtained more regularity results in higher dimensions on general domains. Let  $f$  satisfy (H) and define

$$\tau_- := \liminf_{t \rightarrow a_f} \frac{f(t)f''(t)}{f'(t)^2} \leq \tau_+ := \limsup_{t \rightarrow a_f} \frac{f(t)f''(t)}{f'(t)^2}. \quad (1.3)$$

In [6] the authors also show that for a regular and superlinear nonlinearity  $f$  with  $\tau_- > 0$ ,  $u^*$  is bounded for  $n \leq 7$  (see [6], Theorem 4.1). As we shall see here in Corollary 2.4, with a minor change in their proof, the same holds with a weaker condition. Also, they showed that if  $\tau_+ < \infty$  then  $u^*$  is bounded for  $n < \frac{8}{\tau_+}$ , see Theorem 5.1 in [6].

The main results of this paper are as follows.

**Theorem 1.1.** *Let  $f$  satisfy (H) with  $0 < \tau_- \leq \tau_+ < 2$ , and  $\Omega$  an arbitrary bounded smooth domain. Also, let  $u_m$  be a sequence of semistable solutions of  $(N_{\lambda_m})$  satisfy the stability inequality (1.2). Then  $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$  for*

$$n < N(f) := \begin{cases} \frac{4\alpha_*(2-\tau_+)+2\tau_+}{\tau_+} \max\{1, \tau_+\} & f \text{ is regular,} \\ \frac{4\alpha_*(2-\tau_+)+2\tau_+}{\tau_+} & f \text{ is singular,} \end{cases} \quad (1.4)$$

where  $\alpha_* > 1$  denotes the largest root of the polynomial

$$P_f(\alpha, \tau_-, \tau_+) := (2 - \tau_-)^2 \alpha^4 - 8(2 - \tau_+) \alpha^2 + 4(4 - 3\tau_+) \alpha - 4(1 - \tau_+). \quad (1.5)$$

As a consequence,

if  $\tau_- = \tau_+ := \tau$ , then  $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$  for  $n \leq 12$  when  $\tau \leq 1$ , and for  $n \leq 7$  when  $\tau \leq 1.57863$ .

**Corollary 1.2.** *Let  $f$  satisfy (H) be a regular nonlinearity with  $0 < \tau_- \leq \tau_+ < 2$  and  $\Omega$  an arbitrary bounded smooth domain. Let  $u^*$  be the extremal solution of problem  $(N_\lambda)$ . Then  $u^* \in L^\infty(\Omega)$  for*

$$n < \frac{4\alpha_*(2 - \tau_+) + 2\tau_+}{\tau_+} \max\{1, \tau_+\}.$$

In particular, if  $\tau_- = \tau_+$  then  $u^* \in L^\infty(\Omega)$  for  $n \leq 12$ .

For example consider problem (1.1) with  $f(u) = e^u$  or  $e^{u^\alpha}$  ( $\alpha > 0$ ), then  $\tau_+ = \tau_- = 1$ , hence by Theorem 1.1,  $u^* \in L^\infty(\Omega)$  for  $n \leq 12$ . The same is true for  $f(u) = (1+u)^p$  ( $p > 1$ ) as in this case we have  $\tau_+ = \tau_- = \frac{p-1}{p}$ . More precisely we have  $u^* \in L^\infty(\Omega)$  for  $n < \frac{4(p+1)}{p-1}\alpha_* + 2$  where  $\alpha_*$  denotes the largest root of the polynomial

$$P_f(\alpha) := (p+1)^2\alpha^4 - 8p(p+1)\alpha^2 + 4p(p+3)\alpha - 4p. \quad (1.6)$$

This is exactly the same as the result obtained by Hajlaoui-Harrabi-Ye in [18].

Now consider problem (1.1) with the singular nonlinearity  $f(u) = (1-u)^{-p}$  ( $p > 1$ ) and  $\Omega$  an arbitrary bounded smooth domain. Then from the fact that  $\tau_+ = \tau_- = \frac{p+1}{p}$  and Theorem 1.1, we get  $\|u^*\|_{L^\infty(\Omega)} < 1$  for  $n < 4\alpha_* + 2$ , where  $\alpha_*$  denotes the largest root of the polynomial

$$P_f(\alpha) := \alpha^4 - 8\frac{p(p-1)}{(p+1)^2}\alpha^2 + 4\frac{p(p-1)(p-3)}{(p+1)^3}\alpha + 4\frac{p(p-1)^2}{(p+1)^4}. \quad (1.7)$$

This results coincides with that of Guo-Wei [17]. In particular, when  $p > 1.72822$  then  $\|u^*\|_{L^\infty(\Omega)} < 1$  for  $n \leq 7$ . Also, when  $p > 2.2609$  the same is true for  $n \leq 8$ .

## 2. Preliminaries and Auxiliary Results

The following standard regularity result is taken from [8], for the proof see Theorem 3 of [19].

**Proposition 2.1.** *Let  $u \in H_0^1(\Omega)$  be a weak solution of*

$$\begin{cases} \Delta u + c(x)u = g(x) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (2.1)$$

*with  $c, g \in L^q(\Omega)$  for some  $q > \frac{n}{2}$ . Then there exists a positive constant  $C$  independent of  $u$  such that:*

$$\|u\|_{L^\infty(\Omega)} \leq C(\|u\|_{L^1(\Omega)} + \|g\|_{L^p(\Omega)}). \quad (2.2)$$

Consider problem  $(N_\lambda)$ . By the elliptic regularity we know that, if for some  $q \geq 1$  we have  $\|f(u_\lambda)\|_{L^q(\Omega)} \leq C$ , where  $C$  is a constant independent of  $\lambda$ , then  $u^*$  is bounded, (hence smooth when  $f$  is regular), whenever  $n < 4q$ . Using the above proposition we show that, a similar result holds (for regular or singular nonlinearity) if  $f'(u_\lambda)$  is uniformly bounded in  $L^q(\Omega)$ . For the proof we need the following two lemmas, the first one gives pointwise estimate on  $\Delta u$  for a solution  $u$  of problem  $(N_\lambda)$ , for the proof see [6].

Define the functions  $F, g, \tilde{f} : [0, a_f) \rightarrow \mathbb{R}$  as

$$F(t) = \int_0^t f(s)ds, \quad g(t) = \sqrt{2}(F(t)-t)^{\frac{1}{2}} \quad \text{and} \quad \tilde{f}(t) = f(t)-f(0), \quad 0 \leq t < a_f. \quad (2.3)$$

**Lemma 2.1.** (Lemma 3.2 [6]) Let  $u$  be a solution of problem (1.1). Then

$$-\Delta u \geq \sqrt{\lambda}g(u), \quad \text{in } \Omega.$$

**Lemma 2.2.** Let  $u$  be a semistable solution of problem  $(N_\lambda)$  with  $f$  satisfy (H). Then

$$\int_{\Omega} -\Delta u dx < C,$$

where  $C$  is a constant independent of  $u$ .

PROOF. Let  $\psi$  be the unique positive smooth function such that

$$\begin{cases} -\Delta \psi = 1 & x \in \Omega, \\ \psi = 0 & x \in \partial\Omega, \end{cases}$$

Let  $u$  be a semistable solution of problem  $(N_\lambda)$ . By multiplying the equation  $\Delta^2 u = \lambda f(u)$  in  $\psi$  and then an integration we get (using Green's formula)

$$\lambda \int_{\Omega} \psi(x) f(u) dx = \int_{\Omega} \psi(x) \Delta^2 u dx = \int_{\Omega} \Delta \psi(x) \Delta u dx = \int_{\Omega} -\Delta u dx.$$

This gives that

$$\int_{\Omega} -\Delta u dx \leq \lambda \max_{\Omega} \psi(x) \int_{\Omega} f(u) dx.$$

The inequality above and the uniform  $L^1(\Omega)$  boundedness of  $f(u)$  for semistable solutions (proved in Lemma 3.5 in [6]) gives the desired result. ■

In the sequel we will frequently use the following simple lemma.

**Lemma 2.3.** Let  $g_1, g_2 : [0, a_f) \rightarrow [0, \infty)$  be continuous functions such that for some  $T \in (0, a_f)$  we have  $g_2(t) \leq g_1(t)$ ,  $T \leq t < a_f$ . If for a sequence  $u_m$  of solutions of problem  $N_{\lambda_m}$  we have

$$\int_{\Omega} g_1(u_m) dx \leq C,$$

where  $C$  is a constant independent of  $u_m$ , then the same holds for  $\int_{\Omega} g_2(u_m) dx$ .

PROOF. Indeed, we have

$$\begin{aligned} \int_{\Omega} g_2(u_m) dx &= \int_{u_m \leq T} g_2(u_m) dx + \int_{u_m > T} g_2(u_m) dx \leq M|\Omega| + \int_{\Omega} g_1(u_m) dx \\ &\leq M|\Omega| + C, \quad \text{where } M := \sup_{[0, T]} g_2(t). \end{aligned}$$

■

**Proposition 2.2.** *Let  $f$  satisfy (H) (when  $f$  is singular we additionally assume that  $\lim_{t \rightarrow a_f} F(t) = \infty$ ). Let  $u_m$  be a sequence of semistable solutions of problem  $(N_{\lambda_m})$ . If  $\sup_m \|\frac{\tilde{f}(u_m)}{\sqrt{F(u_m)}}\|_{L^q(\Omega)} < \infty$ , for some  $q \geq 1$ , then*

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \quad (2.4)$$

for  $n < 2q$ . In particular, if  $\sup_m \|f'(u_m)\|_{L^q(\Omega)} < \infty$ , then (2.4) holds for  $n < 4q$ .

PROOF. Take  $v_m := -\Delta u_m$ , then from (1.1)  $v_m$  satisfies

$$\begin{cases} \Delta v_m + \lambda_m f(u_m) = 0 & x \in \Omega, \\ v_m = 0 & x \in \partial\Omega. \end{cases} \quad (2.5)$$

We rewrite problem (2.5) as  $\Delta v_m + c_m(x)v_m = -\lambda_m f(0)$  where  $c_m(x) := \lambda_m \frac{\tilde{f}(u_m)}{v}$ . By the pointwise estimate in Lemma 2.1 we have

$$0 \leq c_m(x) = \lambda_m \frac{\tilde{f}(u_m)}{v_m} \leq \sqrt{\lambda_m} \frac{\tilde{f}(u_m)}{g(u_m)},$$

Now using the inequality

$$0 \leq \frac{\tilde{f}(t)}{g(t)} \leq \sqrt{2} \frac{\tilde{f}(t)}{\sqrt{F(t)}}, \quad t > T, \quad \text{for some } T < a_f,$$

which comes from the fact that  $\lim_{t \rightarrow a_f} \frac{g(t)}{\sqrt{2F(t)}} = 1$ , and the assumptions, we get  $\sup_m \|c_m(x)\|_{L^q(\Omega)} < \infty$ . Thus, by the assumptions, Lemma 2.2 and Proposition 2.1,  $\|v_m\|_{L^\infty(\Omega)} \leq C$ , and hence  $\|F(u_m)\|_{L^\infty(\Omega)} \leq C$  (by Lemma 2.1), where  $C$  is a constant independent of  $m$ , for  $n < 2q$ . Now the fact that  $\lim_{t \rightarrow a_f} F(t) = \infty$  gives the first part. To prove the second part, it suffices to use Lemma 2.3 and note that by the convexity of  $f$ , we have

$$\frac{\tilde{f}(t)}{\sqrt{F(t)}} \leq 2\sqrt{f'(t)}, \quad \text{for } t \text{ sufficiently close to } a_f. \quad (2.6)$$

Indeed,  $f'$  is a nondecreasing function by the convexity of  $f$ , thus we have, for  $0 < t < a_f$

$$f'(t)F(t) = f'(t) \int_0^t f(s)ds \geq \int_0^t f'(s)f(s)ds = \frac{f(t)^2}{2} - \frac{f(0)^2}{2},$$

now the fact that  $f(t) \rightarrow \infty$  as  $t \rightarrow a_f$  gives (2.6). ■

**Remark 2.1.** *The condition  $\lim_{t \rightarrow a_f} F(t) = \infty$  in the above proposition, which is needed for a singular nonlinearity  $f$ , is satisfied by the extra assumption that*

$\tau_+ < 2$ . Indeed, for a  $\tau \in (\tau_+, 2)$  there exists  $T \in (0, a_f)$  such that  $\frac{f''(t)}{f'(t)} \leq \tau \frac{f'(t)}{f(t)}$  for  $t \in (T, a_f)$ , thus by an integration we get  $f'(t) \leq C f(t)^\tau$  or equivalently  $f'(t) f(t)^{1-\tau} \leq C f(t)$  for  $t \in (T, a_f)$ . Again an integration gives

$$\frac{f(t)^{2-\tau}}{2-\tau} - \frac{f(T)^{2-\tau}}{2-\tau} \leq C(F(t) - F(T)), \quad \text{for } t \in (T, a_f).$$

Now the facts that  $\lim_{t \rightarrow a_f} f(t) = \infty$  and  $\tau < 2$  imply that  $\lim_{t \rightarrow a_f} F(t) = \infty$ .

For example, take the singular nonlinearity  $f(t) = (1-t)^{-p}$  ( $p > 1$ ) on  $[0, 1)$ . We have  $\tau_- = \frac{p+1}{p} \in (0, 2)$  and

$$F(t) = \frac{1}{p-1} \left( \frac{1}{(1-t)^{p-1}} - 1 \right) \rightarrow \infty, \quad \text{as } t \rightarrow 1.$$

Then, as a corollary of Proposition 2.2, we have the next regularity result for problem  $(N_\lambda)$ . It is proved in [7, 6] by a different proof with the restriction that  $p \neq 3$ .

**Proposition 2.3.** *Let  $f(u) = (1-u)^{-p}$  ( $p > 1$ ) and  $u_m$  be a sequence of semistable solutions of problem  $(P_{\lambda_m})$ , such that for some  $q > 1$  and  $q \geq \frac{(p+1)n}{4p}$  so that  $\sup_m \|f(u_m)\|_{L^q(\Omega)} < \infty$ . Then  $\sup_m \|u_m\|_{L^\infty(\Omega)} < 1$ .*

PROOF. Notice that we have

$$f'(t) = p(1-t)^{-(p+1)} = f(t)^{\frac{p+1}{p}}, \quad t \in [0, 1).$$

Hence, by the assumption  $\sup_m \|f'(u_m)\|_{L^{\frac{qp}{p+1}}(\Omega)} < \infty$ .

As an application of Proposition 2.2, consider problem  $(N_\lambda)$  with a convex nonlinearity  $f$  satisfies  $(H)$  such that  $f(t) = t \ln t$  for  $t$  large. Then, for every  $\epsilon > 0$  there exist  $T_\epsilon, C_\epsilon > 0$  such that  $f'(t) \leq f(t)^\epsilon$  for  $t \geq T_\epsilon$ .

$$f'(t) \leq f(t)^\epsilon \quad \text{for } t \geq T_\epsilon. \quad (2.7)$$

Now if  $u \geq 0$  is a semistable solution of problem (1.1), from Lemma 3.5 [6] we have  $\int_\Omega f(u) dx \leq C$  with  $C$  independent of  $\lambda$  and  $u$ . This together (2.7) and Lemma 2.3 give  $f'(u) \in L^{\frac{1}{\epsilon}}(\Omega)$  uniformly, hence by Proposition 2.2,  $u^*$  is bounded for  $n < \frac{4}{\epsilon}$ , and since  $\epsilon > 0$  was arbitrary,  $u^*$  is bounded in every dimension  $n$ . Indeed, the same result is true for every regular nonlinearity  $f$  satisfies  $(H)$  with  $\tau_+ = 0$  or equivalently

$$\lim_{t \rightarrow \infty} \frac{f(t)f''(t)}{f'(t)^2} = 0. \quad (2.8)$$

Indeed, (2.8) implies (2.7) and we can proceed as above.

The following lemma is a special case of an interesting result of [6].

**Lemma 2.4.** *Let  $u$  be a semistable solution of problem  $(N_\lambda)$ . If  $H(t) := \int_0^t f''(s)\sqrt{F(s)}ds$  for  $t \geq 0$ . Then*

$$\int_{\Omega} \sqrt{F(u)}H(u)dx \leq C, \quad (2.9)$$

where  $C$  is a constant independent of  $\lambda$  and  $u$ .

When  $f$  is regular, in [6] the authors used the above lemma to prove that  $u^*$  is bounded for  $n < \frac{8}{\tau_+}$ . In a completely similar manner and using Proposition 2.2, we can prove a similar result when  $f$  is singular.

**Lemma 2.5.** *Let  $f$  satisfy (H) be a singular nonlinearity with  $0 < \tau_+ < 2$ , and  $u_m$  be a sequence of semistable solutions of problem  $(N_{\lambda_m})$ . Then*

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \quad (2.10)$$

for  $n < \frac{8}{\tau_+}$ .

PROOF. Take an arbitrary number  $\tau > \tau_+$ , then from the definition of  $\tau_+$  there exists a  $T_1 \in (0, a_f)$  such that  $\frac{f(t)f''(t)}{f'(t)^2} \leq \tau$ ,  $T_1 \leq t < a_f$ , which is equivalent to  $\frac{d}{dt}(\frac{f'(t)}{f(t)^\tau}) \leq 0$  for  $T_1 \leq t < a_f$ . This gives  $f'(t) \leq C_0 f(t)^\tau$  for  $T_1 \leq t < a_f$ . Hence, using the inequality (2.6),  $F(t) \geq C_1 f'(t)^{\frac{2}{\tau}-1}$ ,  $T_1 \leq t < a_f$ , for some  $T_2 \in (T_1, a_f)$ . Thus, for a  $T > T_2$  sufficiently close to  $a_f$  we have

$$\begin{aligned} \sqrt{F(t)}H(t) &\geq C_2 f'(t)^{\frac{1}{\tau}-\frac{1}{2}} \int_T^t f''(s)f'(s)^{\frac{1}{\tau}-\frac{1}{2}} ds \\ &\geq C_3 f'(t)^{\frac{2}{\tau}}, \text{ for } t > T \text{ sufficiently close to } a_f. \end{aligned}$$

Using the inequality above, Lemma 2.3 and Lemma 2.4, we have  $\|f'(u_m)\|_{L^{\frac{2}{\tau}}(\Omega)} \leq C$ . Hence by Remark 2.1 and Proposition 2.2,  $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$  for  $n < \frac{8}{\tau}$ , and since  $\tau > \tau_+$  was arbitrary we get (2.10). ■

As we have mentioned before, another main result of [6] is that if  $\tau_- > 0$  then  $u^*$  is bounded for  $n \leq 7$ . Using the same proof of this in [6] we can prove it by a weaker assumption as follows:

**Corollary 2.6.** *Consider problem  $(N_\lambda)$  with a regular nonlinearity  $f$  satisfies (H) such that for some  $0 \leq \epsilon < 1$*

$$\liminf_{t \rightarrow \infty} \frac{f(t)^{1+\frac{\epsilon}{4}} f''(t)}{f'(t)^2} > 0. \quad (2.11)$$

Then  $u^*$  is bounded for  $n \leq 7$ .



PROOF. From (2.11) we have  $f(t)^{1+\frac{\epsilon}{4}}f''(t) \geq C_0f'(t)^2$ ,  $t \geq T$ , for some  $T > 0$ . Hence, using the inequality (2.6) and the fact that  $F$  is a nondecreasing function we get, for a  $T' > T$  sufficiently large,

$$\begin{aligned}\sqrt{F(t)}H(t) &\geq C_1 \int_0^t f''(s)F(s)ds \geq C_2 \int_T^t \frac{f''(s)f(s)^2}{f'(s)} \geq C_3 \int_T^t f'(s)f(s)^{1-\frac{\epsilon}{4}} \\ &\geq C_4 f(t)^{2-\frac{\epsilon}{4}} \text{ for } t > T'.\end{aligned}$$

Thus, from Lemmas 2.3 and 2.4 we have  $\|f(u)\|_{L^{2-\frac{\epsilon}{4}}(\Omega)} < C$  where  $C$  is independent of  $u$ . Now the elliptic regularity implies  $u^*$  is bounded for  $n \leq 8 - \epsilon > 7$ , that gives the desired result. ■

### 3. Proof of the main results

Following the idea of Dupaigne, Ghergu and Warnault in [7], we prove the following lemma which is crucial for the proof of the main results.

**Lemma 3.1.** *Let  $u$  be a positive smooth solution of  $(N_\lambda)$  satisfy the stability inequality (1.2),  $\theta : [0, a_f) \rightarrow [0, \infty)$  a  $C^1$  positive function with  $\theta(0) = 0$ , and  $\Theta(t) := \int_0^t \theta'(s)^2 ds$ , for  $0 \leq t < a_f$ . Then for every  $\alpha > \frac{1}{2}$  we have*

$$\int_{\Omega} \sqrt{f'(u)}\theta(u)^2 dx \leq \frac{\alpha^2}{2\alpha-1} \left( \int_{\Omega} \frac{f(u)^{2\alpha}}{f'(u)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{1}{2\alpha}} \left( \int_{\Omega} \frac{\Theta(u)^{\frac{2\alpha}{2\alpha-1}}}{f'(u)^{\frac{1}{2(2\alpha-1)}}} dx \right)^{\frac{2\alpha-1}{2\alpha}}. \quad (3.1)$$

PROOF. Let  $u$  be a positive smooth solution of  $(N_\lambda)$  satisfy (1.2) and set  $v := -\Delta u$ . Up to rescaling, we may assume that  $\lambda = 1$ . Take  $\phi = \theta(u)$  as a test function in the stability inequality (1.2). Then we get

$$\int_{\Omega} \sqrt{f'(u)}\theta(u)^2 dx \leq \int_{\Omega} v\theta(u) dx. \quad (3.2)$$

Also, taking  $\phi = v^\alpha$  ( $\alpha > \frac{1}{2}$ ) as a test function in the stability inequality (1.2), we get

$$\int_{\Omega} \sqrt{f'(u)}v^{2\alpha} dx \leq \frac{\alpha^2}{2\alpha-1} \int_{\Omega} f(u)v^{2\alpha-1} dx. \quad (3.3)$$

Using Hölder inequality (with two conjugate numbers  $2\alpha$  and  $\frac{2\alpha}{2\alpha-1}$ ) on the right-hand side of inequality (3.2) we get

$$\int_{\Omega} \sqrt{f'(u)}\theta(u)^2 dx \leq \left( \int_{\Omega} \sqrt{f'(u)}v^{2\alpha} dx \right)^{\frac{1}{2\alpha}} \left( \int_{\Omega} \frac{\Theta(u)^{\frac{2\alpha}{2\alpha-1}}}{f'(u)^{\frac{1}{2(2\alpha-1)}}} dx \right)^{\frac{2\alpha-1}{2\alpha}}. \quad (3.4)$$

Similarly, from (3.3) and Hölder inequality we get

$$\int_{\Omega} \sqrt{f'(u)}v^{2\alpha} dx \leq \frac{\alpha^2}{2\alpha-1} \left( \int_{\Omega} \sqrt{f'(u)}v^{2\alpha} dx \right)^{\frac{2\alpha-1}{2\alpha}} \left( \int_{\Omega} \frac{f(u)^{2\alpha}}{f'(u)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{1}{2\alpha}},$$

that gives

$$\int_{\Omega} \sqrt{f'(u)} v^{2\alpha} dx \leq \left( \frac{\alpha^2}{2\alpha-1} \right)^{2\alpha} \int_{\Omega} \frac{f(u)^{2\alpha}}{f'(u)^{\alpha-\frac{1}{2}}} dx. \quad (3.5)$$

Plugging (3.5) in (3.4) we arrive at

$$\int_{\Omega} \sqrt{f'(u)} \theta(u)^2 dx \leq \frac{\alpha^2}{2\alpha-1} \left( \int_{\Omega} \frac{f(u)^{2\alpha}}{f'(u)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{1}{2\alpha}} \left( \int_{\Omega} \frac{\Theta(u)^{\frac{2\alpha}{2\alpha-1}}}{f'(u)^{\frac{1}{2(2\alpha-1)}}} dx \right)^{\frac{2\alpha-1}{2\alpha}},$$

which is the desired result. ■

### Proof of Theorem 1.1

Fix an  $\alpha > 1$  such that  $P_f(\alpha, \tau_-, \tau_+) < 0$ . Such an  $\alpha$  exists since we have  $P_f(1, \tau_-, \tau_+) = (2 - \tau_-)^2 - 4 < 0$  and  $P_f(+\infty, \tau_-, \tau_+) = +\infty$ . Now take positive numbers  $\tau_1 \in (0, \tau_-)$  and  $\tau_2 \in (\tau_+, 2)$  such that

$$P_f(\alpha, \tau_1, \tau_2) < 0. \quad (3.6)$$

We claim that

$$I_m := \int_{\Omega} \frac{\tilde{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx < C, \quad (3.7)$$

where  $C$  is independent of  $m$ . To this end, take  $\theta(t) = \frac{\tilde{f}(t)^\alpha}{f'(t)^{\frac{\alpha}{2}}}$  in the inequality (3.1).

First we estimate the function  $\Theta(t) = \int_0^t \theta'(s)^2 ds$  as follows. We have

$$\Theta(t) = \alpha^2 \int_0^t \tilde{f}(s)^{2\alpha-2} f'(s)^{2-\alpha} \left( 1 - \frac{\tilde{f}(s)f''(s)}{2f'(s)^2} \right)^2 ds. \quad (3.8)$$

By the definitions of  $\tau_{\pm}$  there exists a  $T < a_f$  such that  $\tau_1 \leq \frac{\tilde{f}(t)f''(t)}{f'(t)^2} \leq \tau_2$  for  $T \leq t < a_f$  that also gives

$$0 < 1 - \frac{\tau_2}{2} \leq \frac{\tilde{f}(t)f''(t)}{2f'(t)^2} \leq 1 - \frac{\tau_1}{2}, \quad \text{for } T \leq t < a_f. \quad (3.9)$$

Using (3.9) in (3.8) we get

$$\Theta(t) \leq \Theta(T) + \alpha^2 \left( 1 - \frac{\tau_1}{2} \right)^2 \int_T^t \tilde{f}(s)^{2\alpha-2} f'(s)^{2-\alpha} ds, \quad \text{for } T \leq t < a_f. \quad (3.10)$$

Now, notice that taking  $h(t) := \tilde{f}(t)^{2\alpha-1} f'(t)^{1-\alpha}$  for  $0 \leq t < a_f$ , then

$$\begin{aligned} h'(t) &= (2\alpha-1) \tilde{f}(t)^{2\alpha-2} f'(t)^{2-\alpha} \left( 1 - \frac{\alpha-1}{2\alpha-1} \frac{\tilde{f}(s)f''(s)}{f'(s)^2} \right) \\ &\geq (2\alpha-1) \left( 1 - \frac{\alpha-1}{2\alpha-1} \tau_2 \right) \tilde{f}(t)^{2\alpha-2} f'(t)^{2-\alpha}, \quad \text{for } T \leq t < a_f. \end{aligned}$$

Using the above inequality in (3.10) we obtain

$$\Theta(t) \leq C + A\tilde{f}(t)^{2\alpha-1}f'(t)^{1-\alpha}, \text{ where } A := \frac{\alpha^2}{(2\alpha-1)} \frac{(1-\frac{\tau_1}{2})^2}{(1-\frac{\alpha-1}{2\alpha-1}\tau_2)} \text{ and } C := \Theta(T) - Ah(T). \quad (3.11)$$

Note that in the above we also used that  $1 - \frac{\alpha-1}{2\alpha-1}\tau_2 > 0$  which holds since  $\tau_2 < 2$ .

Now, the fact that the inequality  $\frac{\tilde{f}(t)f''(t)}{f'(t)^2} \leq \tau_2$  for  $T \leq t < a_f$  is equivalent to  $\frac{d}{dt}(\frac{f'(t)}{f(t)^{\tau_2}}) \leq 0$  for  $T \leq t < a_f$  gives

$$f'(t) \leq C_1 \tilde{f}(t)^{\tau_2} \text{ for } T \leq t < a_f. \quad (3.12)$$

Using this we obtain, for  $T \leq t < a_f$

$$\tilde{f}(t)^{2\alpha-1}f'(t)^{1-\alpha} \geq f'(t)^{\frac{2\alpha-1}{\tau_2}-(\alpha-1)} \rightarrow \infty, \text{ as } t \rightarrow a_f.$$

Now take an  $\epsilon > 0$ . From the inequality above and (3.11), there exists an  $M_\epsilon \in [T, a_f)$  such that

$$\Theta(t) \leq (A + \epsilon)\tilde{f}(t)^{2\alpha-1}f'(t)^{1-\alpha}, \text{ for } t \in [M_\epsilon, a_f). \quad (3.13)$$

Hence,

$$\frac{\Theta(t)^{\frac{2\alpha}{2\alpha-1}}}{f'(t)^{\frac{1}{2(2\alpha-1)}}} \leq (A + \epsilon)^{\frac{2\alpha}{2\alpha-1}} \frac{\tilde{f}(t)^{2\alpha}}{f'(t)^\alpha}, \text{ for } t \in [M_\epsilon, a_f). \quad (3.14)$$

Also, we can find an  $M'_\epsilon > 0$  such that

$$f(t) \leq (1 + \epsilon)\tilde{f}(t), \text{ for } t \in [M'_\epsilon, a_f). \quad (3.15)$$

Now, taking  $M''_\epsilon := \max\{M_\epsilon, M'_\epsilon\}$ , then plugging (3.15), (3.14) in (3.1) we arrive at

$$\begin{aligned} I_m &= \int_{\Omega} \frac{\tilde{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx \\ &\leq \frac{\alpha^2}{2\alpha-1} \left( C_{\epsilon,m} + (1+\epsilon)^{2\alpha} \int_{u_m \geq M''_\epsilon} \frac{\tilde{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{1}{2\alpha}} \left( C'_{\epsilon,m} + (A+\epsilon)^{\frac{2\alpha}{2\alpha-1}} \int_{u_m \geq M''_\epsilon} \frac{\tilde{f}(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx \right)^{\frac{2\alpha-1}{2\alpha}}, \end{aligned}$$

where

$$C_{\epsilon,m} := \int_{u < M''_\epsilon} \frac{f(u_m)^{2\alpha}}{f'(u_m)^{\alpha-\frac{1}{2}}} dx, \text{ and } C'_{\epsilon,m} := \int_{u_m < M''_\epsilon} \frac{\Theta(u_m)^{\frac{2\alpha}{2\alpha-1}}}{f'(u_m)^{\frac{1}{2(2\alpha-1)}}} dx.$$

Note that  $C_{\epsilon,m}$  and  $C'_{\epsilon,m}$  are bounded by a constant independent of  $m$ . Replacing the integrals on the right-hand side of the above inequality with integrals over the full region  $\Omega$  we get

$$I_m \leq \frac{\alpha^2}{2\alpha-1} \left( C_{\epsilon,m} + (1+\epsilon)^{2\alpha} I_m \right)^{\frac{1}{2\alpha}} \left( C'_{\epsilon,m} + (A+\epsilon)^{\frac{2\alpha}{2\alpha-1}} I_m \right)^{\frac{2\alpha-1}{2\alpha}}. \quad (3.16)$$

Now if (3.7) does not hold then,  $I_m \rightarrow \infty$  as  $m \rightarrow \infty$ . Hence, dividing two sides of (3.16) by  $I_m$  and letting  $m \rightarrow \infty$ , we must have

$$1 \leq \frac{\alpha^2}{2\alpha-1}(1+\epsilon)(A+\epsilon),$$

and since  $\epsilon > 0$  was arbitrary we get

$$1 \leq \frac{\alpha^2}{2\alpha-1}A = \frac{\alpha^4}{(2\alpha-1)^2} \frac{(1-\frac{\tau_1}{2})^2}{(1-\frac{\alpha-1}{2\alpha-1}\tau_2)},$$

which is equivalent to  $P_f(\alpha, \tau_1, \tau_2) \geq 0$ , a contradiction, that proves (3.7). Now, using inequality (3.12) we have

$$\frac{\tilde{f}(t)^{2\alpha}}{f'(t)^{\alpha-\frac{1}{2}}} \geq C_2 f'(t)^{\alpha(\frac{2}{\tau_2}-1)+\frac{1}{2}}, \quad \text{for } T \leq t < a_f,$$

hence, thanks to Lemma 2.3 and (3.7) we get

$$\|f'(u_m)\|_{L^{q_1}(\Omega)} \leq C, \quad \text{where } q_1 := \alpha(\frac{2}{\tau_2}-1) + \frac{1}{2},$$

and  $C$  is a constant independent of  $m$ . Now Proposition 2.2 implies that

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \quad \text{for } n < 4\alpha(\frac{2}{\tau_2}-1) + 2. \quad (3.17)$$

Again from inequality (3.12) we have

$$\frac{\tilde{f}(t)^{2\alpha}}{f'(t)^{\alpha-\frac{1}{2}}} \geq C_3 f(t)^{\alpha(2-\tau_2)+\frac{\tau_2}{2}}, \quad \text{for } T \leq t < a_f,$$

and using the above inequality, Lemma 2.3 and (3.7) we get

$$\|f(u_m)\|_{L^{q_2}(\Omega)} \leq C, \quad \text{where } q_2 := \alpha(2-\tau_2) + \frac{\tau_2}{2},$$

and  $C$  is a constant independent of  $m$ . Hence, in the case when  $f$  is regular, by the elliptic regularity theory

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < \infty, \quad \text{for } n < 4\alpha(2-\tau_2) + 2\tau_2. \quad (3.18)$$

Now, since we can choose  $\tau_2$  arbitrary close to  $\tau_+$  and  $\alpha$  near to the largest root of the polynomial  $P_f$ , then (3.17) and (3.18) complete the proof of the first part.

To see the second part, suppose that  $\tau_- = \tau_+ := \tau > 0$ . If  $\tau < \frac{2}{3}$  then from Lemma 2.5,  $\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f$  for  $n \leq \frac{8}{\tau} > 12$ , so we need to prove it for the case  $\frac{2}{3} \leq \tau \leq 1$ . It is not hard to see (for example by using a computing device) that,

for  $\alpha = \frac{5\tau}{2(2-\tau)}$  we have  $P_f(\alpha, \tau, \tau) < 0$  on the interval  $[\frac{2}{3}, 1]$ , hence  $\alpha^* > \frac{5\tau}{2(2-\tau)}$  that gives

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \quad \text{for } n < 4\alpha^*\left(\frac{2}{\tau} - 1\right) + 2 > 12.$$

Also, when  $1 \leq \tau \leq 1.57863$  then for  $\alpha = \frac{5\tau}{4(2-\tau)}$  we have  $P_f(\alpha, \tau, \tau) < 0$  on the interval  $[\frac{2}{3}, 1]$ , hence  $\alpha^* > \frac{5\tau}{4(2-\tau)}$  that gives

$$\sup_m \|u_m\|_{L^\infty(\Omega)} < a_f, \quad \text{for } n < 4\alpha^*\left(\frac{2}{\tau} - 1\right) + 2 > 7,$$

and now the proof is complete. ■

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